Discrete Maths-05

21/08/24

⇒ Let ρ be a relation in a set A

graph-definition

$\rho \subseteq A \times A$

$(a,b) \in \rho \Leftrightarrow a \rho b$

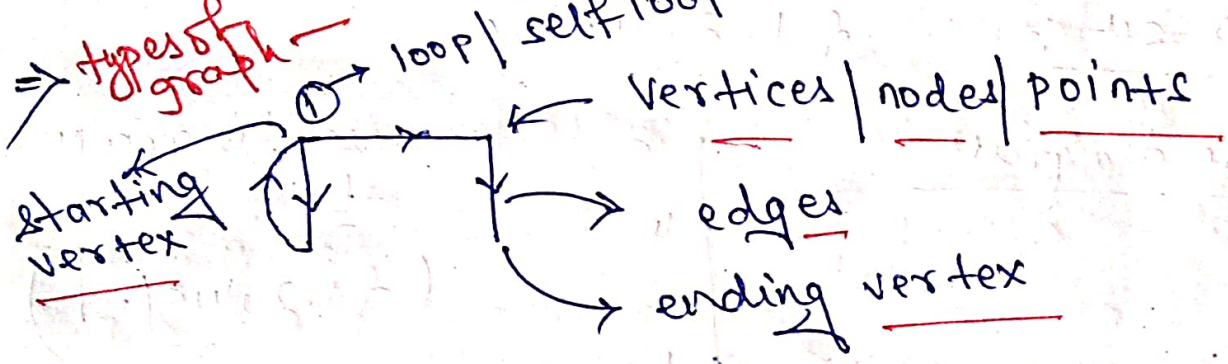
$A = \{1,2,3,4\}$ ✓

$\rho = \{(1,1),(1,2),(1,4),(4,1),(2,3)\}$

	1	2	3	4
1	✓	✓	x	✓
2	x	x	✓	x
3	x	x	x	x
4	✓	x	x	x

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

⇒ types of graph



Directed graph

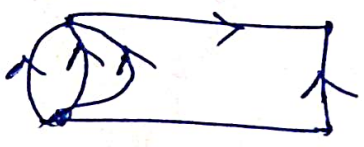
⇒ $\{1,2,3\}$

$\{1,1,2,2,2,3,3\}$

$= \{1^2, 2^3, 3^2\}$

* Multiset → group.

* Multigraph
* Multiedges



* PO set :- ⟶ $S = \{1, 2, 4, 6, 10, 12\}$

$\Downarrow$

$(S, /)$

divisibility

a/b

if a divides b

⇒ Hasse diagram :-



Transitivity

$1/2$

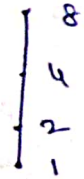
$(1,2) \in /$

Totally ordered set

$S = \{1, 2, 4, 8\}$ every 2 elements is comparable.

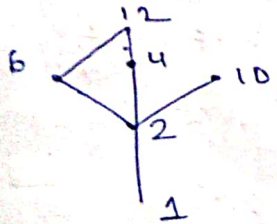
e.g. PO set :- $\{1, 2, 4, 8\}$

Hasse diagram



Hasse diagram of $(\{1, 2, 4, 8\}, /)$

e.g. $\{1, 2, 4, 6, 10, 12\}$



Hasse diagram

e.g. set A

$(\wp(A), \subseteq)$ ⟶ PO set.

⇒ $A = \{1, 2, 3\}$

$\wp(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

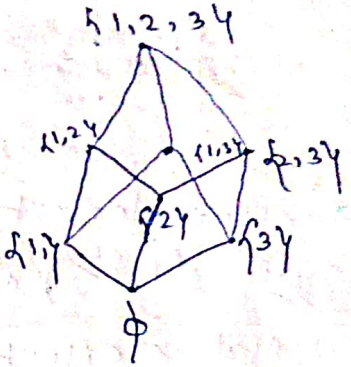
a is a subset of b

b " " " " a

$a \subseteq b$

$a \subseteq b \quad b \subseteq c$

$a \subseteq c$



⇒ Hasse diagram :- ⟶

of $A : (\wp(A), \subseteq$

if totally ordered set then it will just be a line

* Hasse diagram is used to represent PO set, i.e. it will be represent to. or. set.

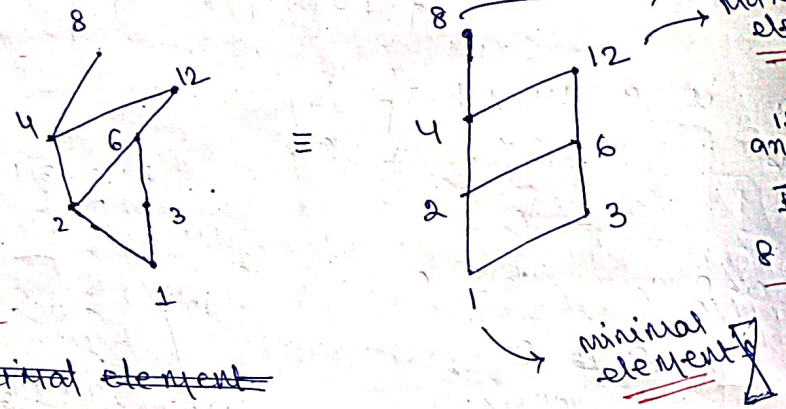
* Lattice ↓ next class

Discrete Maths-06

28/08/24

a) Draw the Hasse diagram of the poset $(\{1,2,3,4,6,8,12\}, |)$.

⇒



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~~Maximal element~~

Let, $(P, \leq)$ be a poset.

Maximal element:

If there is any element $a \in P$ s.t $a < b$ for ~~no~~ ~~any~~ ~~element b in P~~, then 'a' is said to be a maximal element of P.

there is no element $b \in P$ for which

* $A \subseteq B$ and $A \neq B \Rightarrow A \subset B$ → proper subset.

$4 \leq 5$ & $4 \neq 5 \Rightarrow 4 < 5$

$\{1,2\} \subseteq \{1,2,3\}$ & $\{1,2\} \neq \{1,2,3\}$

$\Rightarrow \{1,2\} \subset \{1,2,3\}$ → proper subset.

$\phi \subset A$

$A \subset A$ → except these 2 all other are proper subset.

Minimal element: If there is an element $a \in P$ s.t there is no $b \in P$ for which $b < a$, then a is a minimal element of P.

* **Greatest element**

If there is an element $a \in P$ s.t $x \leq a$ $\forall\, x \in P$ then a is called the greatest element of P, (the poset $\{1,2,3,4,6,8,12\}$ | has no greatest element)

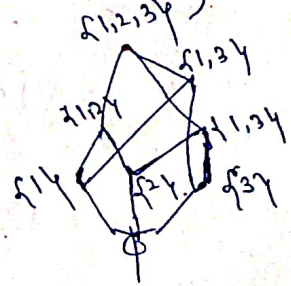
* **least element:-**

If there is an element $a \in P$ s.t $a \leq x$ $\forall\, x \in P$ then a is called the smallest or least element of P.

(the poset $(\{1,2,3,4,6,8,12\}, |)$ then the least element will be 1.)

e.g. $A = \{1,2,3\}$

$(P(A), \subseteq)$



Greatest element : $\{1,2,3\}$.

* upper bound of two elements of a poset.

if there exists $x \in P$ s.t for $a, b \in P$, $a \le x$ & $b \le x$, then x is said to an upper bound of a & b.

6 & 12 are upper bounds of 2 & 3.

in $\{1, 2, 3, 4, 6, 8, 12, 24, \ldots\}$

** along the upper bounds the smallest one is called the smallest / least upper bound / supremum / maximum of those two elements.

⇒ 6 is the supremum of 2 & 3

**

⇒ if $\exists\ x \in P$ s.t for $a \in P$ & $b \in P$, $x \le a$ & $x \le b$, then x is said to a lower bound of a & b.

** greatest lower bound / infimum.

infimum :- the greatest among the set of all lower bounds of two elements of a poset is called the greatest lower bound / infimum of those two elements.

2	4
2	6
2	12
↓ glb	↓ LUB

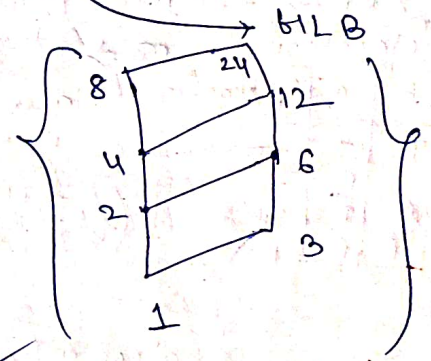
3	4	
↘	↘	
1	12	3, 8
glb	↓ LUB	↓ NO LUB
		1 glb

Imp.

* Lattice :- if there is LUB and glb for any po-set is called the lattice. → (every 2 element in)

* GLB :- if there is '/' → LUB

* LCM :-



{1, 2, 3, 4, 6, 8, 12, 24}

example of lattice

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** $(\mathbb{Z}, /)$ is a lattice, since the glb of any two integers a, b is $\gcd(a, b)$ and lub of a, b is $\text{lcm}(a, b)$.

If there is LUB & GLB for every 2 elem. in po-set then, the po-set is called a lattice.

Discrete Maths-01 (02/04/2024)

Proposition:- A proposition is a declarative sentence which is either true or false but not both.

⇒ eg. why are you late? ✗✗

AND, OR, NOT, XOR.

types of proposins-

⇒ ①

P	~P
T	F
F	T

* AND → conjunction ∧ (Meet)
* OR → disjunction ∨ (Join)
* NOT → Negation ~ (not)
* XOR → exclusive or ⊕
* Implecations ⇒

① Truth table of conjunction :-

True when both propositions are true otherwise false

P	q	P∧q
T	T	T
T	F	F
F	T	F
F	F	F

T → 1
F → 0

(∧ → Meet, × ← AND)

② Truth table of disjunction.

False when both are false & true otherwise

P	q	P∨q
T	T	T
T	F	T
F	T	T
F	F	F

③ Truth table of XOR:

AB' + A'B.

exclusive OR

P	q	P⊕q
T	T	F
T	F	T
F	T	T
F	F	F

XOR: True only when either of the proposition is the true but not both true and false otherwise

④

P	q	P→q	q→P
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

eg.
P: it's raining today
q: I'm carrying an umbrella.
⇒ if 'p' then 'q'.

conditional statement:

if P→q & q→P are same (by TT), then equivalent proposition

*** e.g of necessary & sufficient condition

If even Integer that is div by 2, or an Integer divisible by 2 is even

Now, (P→q) ∧ (q→P) ⇒ P↔q (Biconditional)

P implies & implied by q.

compound

~q	P∧(~q)	T	(P⊕q)	(~P)∨q	~P
F	F	F	T	F	F
T	T	F	F	F	F
F	F	T	F	T	T
T	F		T	T	T

(~P) ✓ (⊕q)

T
F
T
T

Next Page.

VVI

conjunction / disjunction → connectives

in P→q

here false, then write 'F' in (P→q), then check 'q', if F is there, write 'F' in P→q

Q) Find the truth table of $(P \to q) \leftrightarrow (\sim P \vee q)$

P	q	P→q	~P	(~P)∨q	(P→q) ↔ (~P∨q)
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

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tautology

Negation of tautology ⇒ ~ tautology
= contradiction

$(P \to q) \wedge (q \to P) \Rightarrow P \leftrightarrow q$

P	q	~q	P∧(~q)		~(P⊕q)	~P∧q	~P	~P∨q
T	T	F	F	T	T	F	F	T
T	F	T	T	F	F	F	F	F
F	T	F	F	F	F	T	T	T
F	F	T	F	T	T	F	T	T

P	q	P→q	q→P
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

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Discrete Maths-08

9/9/24

Tautology & contradiction

1. Direct proof $A \to B$
2. Indirect proof

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Q) using truth table prove

$A \to B \Leftrightarrow \sim B \to \sim A$

A	B	A→B	~B	~A	~B→~A	A→B ⇔ ~B→~A
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

V.V.V.I

* Rules of Inference :-

1) Modus ponens (law of detachment) — in latin

P
$P \to q$
$\therefore q$

e.g. 20 is even number — P
Even numbers are composite — $P \to q$
∴ 20 is composite. — $\therefore q$

2) Modus tolens :-

$\sim q$
$P \to q$
$\therefore \sim P$

e.g. 5 is not a composite — $\sim q$
Even no.s are 4 — $P \to q$
5 is not even no. — $\therefore \sim P$

4) Transitivity :- (Hypothetical syllogism)

$$\frac{p \to q \quad q \to r}{\therefore\ p \to r}$$

p	q	r	p→q	q→r	p→r	(p→q)∧(q→r)	(p→q)∧(q→r) ↔ p→r
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

p→q, r→q, p→r ✗

5) Disjunctive Syllogism :-

$$\frac{p \vee q \quad \neg p}{\therefore\ q}$$

$\Rightarrow (p \vee q) \wedge (\neg p) \to q$

6) Addition :- $\dfrac{p}{\therefore\ p \vee q}$ $\quad p \to (p \vee q)$

7) Simplification :-

$$\frac{p \wedge q}{\therefore\ p \to \text{false}}$$

20 is a multiple of 5 & a multiple of 10

$\Rightarrow$ 20 is a multiple of 5

7) conjunction :-

$$\frac{p \quad q}{\therefore\ p \wedge q}$$

8) Resolution :-

$$\frac{p \vee q \quad \neg p \vee r}{\therefore\ q \vee r}$$

$(p \vee q) \wedge (\neg p \vee r) \to q \vee r$

p	¬p	p ∨ ¬p
F	T	T
T	F	T

p ∨ ¬p is a tautology ✓

p ∧ ¬p is a contradiction ✓

example. 1.

6 number is even, it's composite
6 number is not comp, it is prime

So the number is either prime/comp.

अर्थात् ∧ या ∨ असल्यास or

If, then असल्यास ¬p or r.

e.g. of resolution

p : 6 is even
q : 6 is prime
r : 6 is multiple of 3

Quantifiers :-

1) Existential quantifier ($\exists$)
read as there exists

2) universal quantifier ($\forall$)
read as for all.

eg. $x > 0 \quad \forall x \in \mathbb{Z}^+$

$\exists x \in \mathbb{Z}, x < 0$

$\exists x \in \mathbb{N}; x < 0$ is F

whereas $\exists x \in \mathbb{Z}, x < 0$ is T.

⇒

i	x_i	y_i
1	1	2
2	2	1
3	5	4
4	6	6

$\sim(x_i > y_i \ \forall i \in \{1,2,3,4\})$

$\exists i \in \{1,2,3,4\}, x_i \not> y_i$

negation of uni. quan. is an ex. quan.

$\sim(x_i y_i < 40 \quad \forall i \in \{1,2,3,4\})$

$x_i y_i \not< 40 \quad \forall i \in \{1,2,3,4\}$ is F

$\Rightarrow \exists i \in \{1,2,3,4\}, x_i y_i \geq 40$

$\sim(x_i y_i < 40 \ \forall i \in \{1,2,3,4\})$

$\exists i \in \{1,2,3,4\}, x_i y_i \geq 40$

$x_i y_i \geq 50 \quad \forall i \in \{1,2,3,4\}$ is false

$\Rightarrow \sim \exists i \in \{1,2,3,4\}, x_i y_i \geq 50$

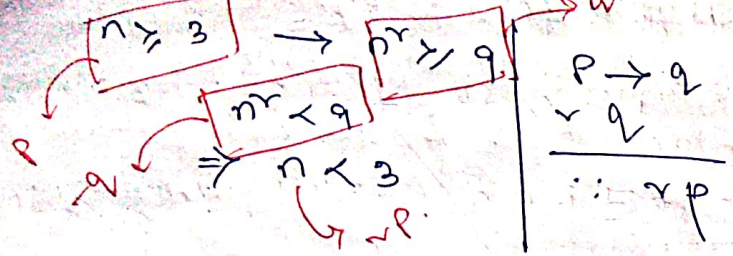
Q) Determine whether the following logical argument is correct. if so, what rules of inferences are used? if not where is the error?

if n is a $\mathbb{R}$ with $n \geq 3$ then $n^2 \geq 9$

suppose $n \not\geq 3$ then $n^2 < 9$ → Question

$\sim n^2 < 9$ then $n < 3$ ✓

⇒ Modus tolens :-



the correct rule of ~~def~~ inference. modus tolens.

$n^r < n$ then $n < 3$

$n < 3$ then $n^r < q$ (Incorrect)

Q) If $x^r \neq 0$ where x is real no. then $x \neq 0$

Let a be a $\mathbb{R}$ with $a^r \neq 0$ then $a \neq 0$

$\Rightarrow$ $\forall \mathbb{R}$ which are $\neq 0$

$\Rightarrow$ $P(x)$ is true $\forall x \in S$

$a \in S$

$\therefore P(a)$

Discrete Maths - 09 (11/9/24)

Q) Does there exist irrational number x & y such that x^y is rational?

$\Rightarrow$ Let $z_1, z_2 \in \mathbb{C}$ → complex no.

Does there exist $z_1, z_2 \in \mathbb{C}$: $z_1^{z_2} \in \mathbb{R}$

$i^i = ?$

$\frac{p}{q}, q \neq 0$

$\frac{\sqrt{3}}{2}$ is irrational

$x = \sqrt{2}, y = \sqrt{2}$

$z = x^y = \sqrt{2}^{\sqrt{2}}$ → if it is rational $(\sqrt{2}, \sqrt{2})$

if $\sqrt{2}^{\sqrt{2}} = r$ is not irrational pair satisfies the property

$z^{\sqrt{2}} = \left((\sqrt{2})^{\sqrt{2}}\right)^{\sqrt{2}}, (\sqrt{2})^2 = 2$

$2^3 = 8$

$8^{\frac{1}{3}} = 2, 2\omega, 2\omega^2$

Q) PT $n^{r+1} \geq 2^n$ for positive integer n such that $1 \leq n \leq 4$

proof of construction

giving exact values → proof of construction

the only possibilities are as follows

n	$n^r + 1$	2^n
1	2	2
2	5	4
3	10	8
4	17	16

Non-constructive existential proof → unable to give exact answer

$\therefore$ we have considered all the cases the proof is done.

Q3) Prove that $n+1 \leq 2^n$ for $n \geq 5, n \in \mathbb{N}$

(Mathematical induction)

for $n=5$, $26 \leq 32$ & so the relation is true.

let, it be true for $n=k$ — non constructive

i.e. $k^r+1 \leq 2^k$

Now,

$(k+1)^r+1$

$= k^r+1+2k+1$

$\leq 2^k+2k+1$, it's less than 2^{k+1}

$\leq 2^k+2k+1 \leq 2^{k+1}$ if $2k+1 \leq 2^k$

$2m+1 \leq 2^m$ for $k \leq 5$

$2(m+1)+1$

$2m+1+2$

$\leq 2m+2$

$\leq 2^m+2^1$ since $m \geq 5 \geq 1$

$\leq 2^{m+1}$

Q4) Prove that there exists two consecutive integers one of which is a perfect sq. and the other is a perfect cube.

let $n = m^r$

& $n+1 = k^3$

then $k^3 - 1 = m^r$

$k^3 = 1 + m^r$

Proof of construction — constructive

Program:-

```
flag = 0
for (k=1; k<=1000; k++).
 for(m=1; m<=1000; m++) {
  if (k*k*k == m*m+1) {
     flag = 1
     exit();
  }
```

Q5) P.T there are no soln's in +ve integers x & y to the equation.

$x^4 + y^4 = 625 = 5^4$

x	y	$x^4+y^4=625$
1	1,2,3,4	
2	1,2,3,4	
3	1,2,3,4	
4	1,2,3,4	
4	4	$256+256=512$

Fermat's last theorem

$x^n + y^n = z^n$ has no integer soln x, y, z with $xyz \neq 0$ & $n > 2$

Q6) can 1729 be written as sum of two perfect cube ? if so in how many ways?

→ 2 way

taxicab number

eg. suppose a & b are odd integers with $a \neq b$ s.t there is a unique integer s.t $|a-c| = |b-c|$

⇒ let $a = 2m+1$
$b = 2n+1$ where $m \neq n$ are integers

then $|a-c| = |b-c|$ gives

either $a-c = b-c$ or $a-c = -(b-c)$

i.e. either $a = b$ or $c = \frac{a+b}{2}$

since $a \neq b$, $c = \frac{a+b}{2} = m+n+1$ a unique integer.

$17+29 = 1^3+12^3 = 9^3+10^3$ ..

A graph is relation on any set X such that $R \subseteq X \times X$

Graph:- relation on a set X

$R \subseteq X \times X$

	1	2	3
1	1	1	0
2	0	0	1
3	0	0	1

$X = \{1,2,3\}$

$X \times X = \{(1,1),(1,2),(1,3) \dots (3,3)\}$

$R = \{(1,1),(1,2),(3,5)\} \subseteq X \times X$

$(2,3)$,



→ Graph / Digraph → directed graph

→ if edge is assigned $E \subseteq V \times V$

unassigned graph:-

→ if edge is not assigned → $E \subseteq V^{(2)}$

→ if nothing is mention

→ if does not assigned anything

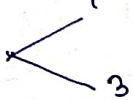
where $V^{(2)}$ is the set of 2 element subsets of V

* It's given by two sets, one set is called set of vertices which is usually non-empty & other set of edges & it's a subset of V.

$V = \{1,2,3\}$

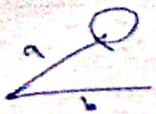
$V^{(2)} = \{\{1,1\},\{1,2\},\{1,3\},\{2,2\},\{2,3\},\{3,3\}\}$

Q) $E = \{\{1,2\},\{2,3\}\}$



three vertices 1,2,3
two edges 1,2 & 2,3

$V = \{1,2,3\}$; $E = \{a,b\}$; $\phi(a) = \{1,2\}$;
$\phi(b) = \{2,3\}$.



$G_1 = V, E, \phi$ → mapping
↓ non empty
↘ if identifies each element of E to pair of element V
→ mapping

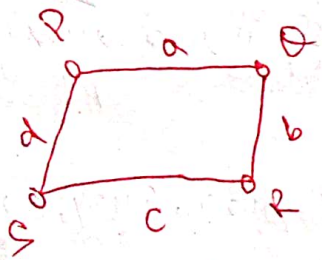
if E is empty then we get NULL graph ✓

$E = \{a,b,c,d\}$

$\phi(a) = \{1,1\}$
$\phi(b) = \{1,2\}$
$\phi(c) = \{1,3\}$
$\phi(d) = \{2,3\}$

* If the end vertices is same then it's self loop
* Nor containing any parallel edges
Nor self loop then it a simple edge.

$V(P,Q,R,S)$
$E(a,b,c,d)$



$\phi(a) = \{P,Q\}$

* Each edge is identified with a pair of vertices through the mapping ϕ, if $a,b \in V$ & $e \in E$ be such that $\phi(e) = \{a,b\}$. then a & b are called end vertices of the edge e.

Fig :



let, $G_1 = (V, E, \phi)$

where, $V = \{a,b,c\}$
$E = \{e,f,g\}$
& $\phi(e) = \{a,b\}$
$\phi(f) = \{a,c\}$
& $\phi(g) = \{b,c\}$

* Adjacent vertices :- if there is an edge b/w two vertices, then they will be called adjacent vertices.

* Incidence :- if a is an end vertex of an edge e, then e will be called incident to a.

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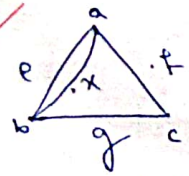
complete graph:- If there is an edge b/w every pair of vertices, i.e. each pair of vertices are adjacent, the graph is called a complete graph.

*(if all the edges are present)

A complete graph with n vertices is denoted by K_n → cardinality

$$|V(K_n)| = n, \quad |E(K_n)| = \frac{n(n-1)}{2}$$

* multi-graph



Multigraph

$E = \{e, f, g, x\}$

$\phi(x) = \{a, b\}$
$\phi(e) = \{a, b\}$] ll edges

* A graph with parallel edges are called a multigraph.

$\{a, b, c\} = \{a, a, b, b, b, c\}$ → Multiset

$= \{a^2, b^3, c\}$

* **Parallel edges:** if two edges have the same pair of vertices, then they will be called parallel edges

* **Self loop:-** If both the end vertices of an edge are identified identical, then that edge is called a self loop.

* **Simple graph:-** A graph with no self loop & parallel edges is called a simple graph.

* **Degree of a vertex:-** the no. of edges incident to a vertex is called the degree of the vertex. denoted by $d(v)$

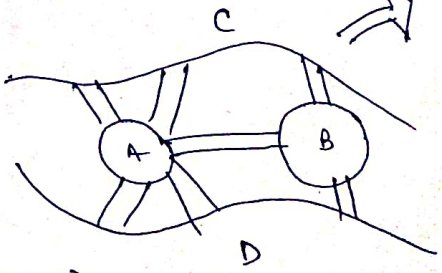
→ in self loop you have to count twice.

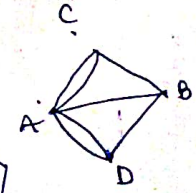
total degree of vertex; will always be even

→ each edge is contributing '2' to total degree of vertex.

(*) **Handshaking lemma:-** for any graph

$$G = (V, E), \quad \sum_{v \in V} d(v) = 2|E|$$





Eulerian graph

* **Graph theory:** innovation — (history)

(Leonard Euler)

- Königsberg's problem

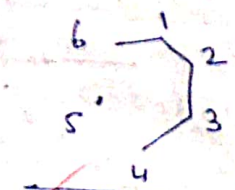
corollary :-

$$\sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = 2|E|$$

where, V_1 : set of even degree vertices

V_2 : " " odd "

No. of odd degree vertices must be even.



odd (~~even~~) no. of vertices has (~~odd~~ even) degree

V.V.I

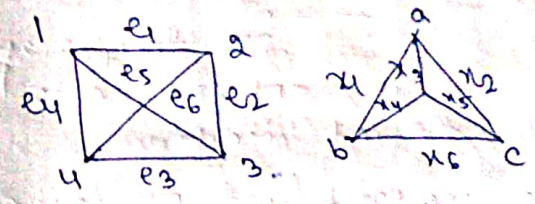
Discrete Maths-12 (25/09/2024)

Graph Isomorphism :- → 1st criteria

V.V.I → Edge (G_1) = Edge (G_2)

→ vertices (G_1) = vertices (G_2)

but if incident relation is not preserved then isomorphism is not possible



$V(G_1) = \{1,2,3,4\}$

$E(G_1) = \{e_1, e_2 \dots e_6\}$

$V(G_2) = \{a,b,c,d\}$

$E(G_2) = \{x_1, x_2 \dots x_6\}$

$1 \leftrightarrow a$	$e_1 \leftrightarrow x_4$
$2 \leftrightarrow b$	$e_2 \leftrightarrow x_6$
$3 \leftrightarrow c$	$e_3 \leftrightarrow x_5$
$4 \leftrightarrow d$	$e_4 \leftrightarrow x_3$
	$e_5 \leftrightarrow x_2$
	$e_6 \leftrightarrow x_1$

⇒ Let $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ be two graphs such that there are one to one correspondance b/w E_1 & E_2 in such a manner that the incidence relation is preserved then G_1 & G_2 will be called isomorphic to each other & is denoted by $G_1 \cong G_2$

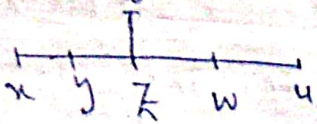
$G_1 \cong G_2$

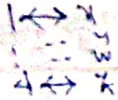
- $\phi : V_1 \to V_2$
- $\psi : E_1 \to E_2$

s.t. if $v_1 \sim v_2$ (adjacent vertices — an edge b/w them) in G_1, then $\phi(v_1) \sim \phi(v_2)$ in G_2 &

$\psi(\{v_1, v_2\}) = \{\phi(v_1), \phi(v_2)\}$

eg: 



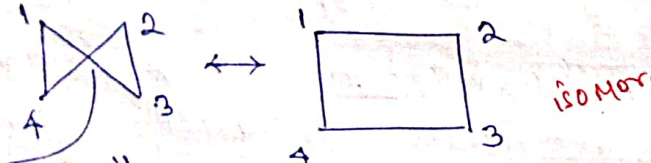


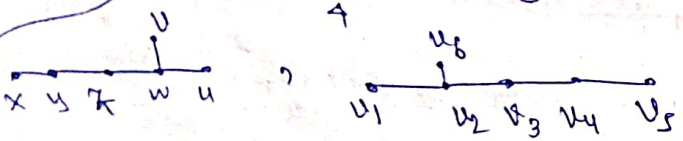
↳ for these graph it's not possible to find bijection

↳ we can't find one to one correspondance b/w edges s.t the incident rel^n is preserved.

* **Intitution:-**

If the two graphs can be redrawn by folding, streching, unfolding - thin those two graphs will be isomorphs

↳ e.g. 

e.g. 

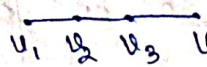
as vertex is not mentioned, then don't consider it as vertex

* **Degree sequence:-** ↳ list containing the degree of all the vertices.

2 graph having equal vertices, equal edges & same degree seq. may or may not isomorphosim

v.v.I **Path Graph**

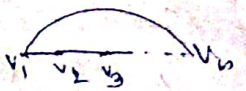
A graph having the vertex set $\{v_1, v_2, \dots v_n\}$ & edges set $E = \{(v_i, v_{i+1}) \mid i = 1, 2, \dots n-1\}$

form:- 

↳ i.e. $v_i \sim v_{i+1}$ for $i = 1, 2 \dots n-1$

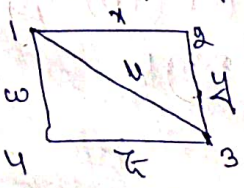
v.v.I **cyclic graph**

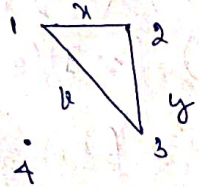
A graph having the vertex set $V = \{v_1, v_2 \dots v_n\}$ & edge set $E = \{(v_i, v_{i+1}) \mid i = 1, 2, \dots n-1\}$, $\cup \{(v_1, v_n)\}$ is called a cyclic graph



v.v.I **subgraph**

let $G_1 = (V_1, E_1)$ & $H = (V_2, E_2)$ be 2 graphs s.t: $V_2 \subseteq V_1$ & $E_2 \subseteq E_1$, then H is called a subgraph of G_1. & denoted as $H \subseteq G_1$.





$V_2 = \{1, 2, 3\} \subseteq V_1$ ✓

$E_2 = \{x, y, z\} \subseteq E_1$ ✗

↳ but (V_2, E_2) is not a graph

$V_1 = \{1, 2, 3, 4\}$ $V_2 = \{1, 2, 3\} \subseteq V_1$

$E_1 = \{x, y, z, w, v\}$ $E_2 = \{x, y, v\} \subseteq E_1$

Spanning subgraph

All the vertices of G_1 is present in G_2 but as for (w) edges in G_2 may / May not be present (means some edges may be missing)

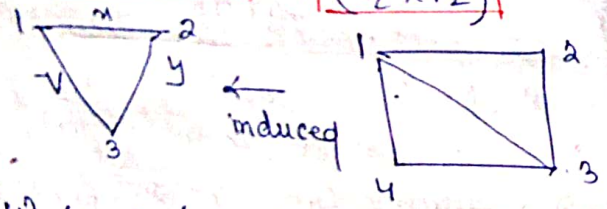
↳ $V_1 = V_2$; $E_1 = E_2$ / $E_1 \neq E_2$

Induced subgraph

It is not necessary to have all the vertices but whatever the vertices you have chosen & edge b/w those vertices should be present

$V_2 \subseteq V_1$ & $E_2 \subseteq E_1 \cap (V_2 \times V_2)$

$V_2 = \{1,2,3\}$

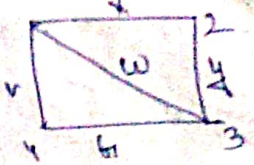


$\{(1,2),(2,3),(3,4),(1,4),(1,3)\} \cap \{(1,1)(1,2),(1,3)$
$(2,1),(2,2),(2,3),(3,1)(3,2)(3,3)\}$
$= \{(1,2)(1,3)(2,3)\}$

walk:- A walk in a graph is an alternating sequence of vertices & edges, starting & ending at vertices : an edge in the seq is incident to the vertices proceeding & succeeding it.

→ start & end in the same vertex called closed walk

→ start & end in another vertex called open walk



vertex ↑ x 2 y 3 w 1 v y z 3

edge

4 z 3 ↓ itself a vertex

e.g. of walk :-

1 x 2 y 3 z 4 v 1 w 3 is a walk in G_1

closed walk: if the starting & ending vertices of a walk are same then

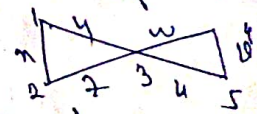
open walk: if the starting & ending vertices are diff

circuit:- A closed walk with no edge repetition is called a ckt.

path:- A path in a graph is an open walk with no vertex repetition.

cycle:- A closed walk with no vertex repetition except the starting & the ending vertex is called a cycle.

1 x 2 z 3 w 4 v 5 u 3 y



Trail:- An open walk with no edge repetition

Eulerian ckt:- A spanning ckt containing all the cycles of a graph is called a Eulerian ckt.

cycle / ckt
path / trail

if a graph contains a Eulerian cycle then that is Eulerian graph.



connected graph :- if there is a path b/w every pair of vertices, then the graph is called a connected graph.

disconnected graph :- A graph which is not connected is called a disconnected graph.

components :- maximal connected sub-graphs of a disconnected graph are called components.



maximal → if you add something to the set the property will not hold anymore.

Discrete Maths-13 (30/9)

Q) V.V.I What is the max no. of edges possible in a simple ~~connected~~ graph with n vertices & k components?

⇒ $\frac{(n-k)(n-k+1)}{2}$ (Ans) → proof at end of Ques-

⇒ Suppose the k-components are comprising of $n_1, n_2, \dots n_k$ number of vertices respectively.

so, $n_1 + n_2 \dots n_k = n$ & the max no. of edges possible in the i^{th} component is $\frac{n_i(n_i-1)}{2}$

Hence, the max no. of edges possible in the whole graph is $\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$

V.V.I Prev. Q.

Q) Max $\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$ s.t $\sum n_i = k$?

⇒ we have $n_i \geq 1$

$n_i - 1 \geq 0$, also ~~sum~~.

$$\sum_{n=1}^{k}(n_i-1) = \sum_{i=1}^{k} n_i - k = n-k$$

⇒ squaring both sides

$$\left\{\sum_{i=1}^{k}(n_i-1)\right\}^2 = (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 + 2\sum_{i=1}^{k}\sum_{j=i+1}^{k}(n_i-1)(n_j-1) = (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 \le (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k}(n_i^2 - 2n_i + 1) \le (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 \le (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2n + k \le (n-k)^2$$

now, $$\sum_{i=1}^{k}\frac{n_i\cdot(n_i-1)}{2} = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2 - n\right)$$

$$\le \frac{1}{2}\left[(n-k)^2 + 2n - k - n\right]$$

$$= \frac{1}{2}\left[(n-k)^2 + 2n - k - n\right]$$

$$= \frac{1}{2}(n-k)(n-k+1)$$

$$(n-1)C_2 = \frac{(n-1)(n-2)}{2} + 1$$

↓ the graph is must be connected

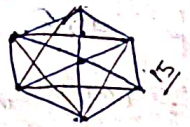
min. ↓ guaranteed edges − 1

corollary :- $\frac{(n-1)(n-2)}{2} + 1$

V.V.I

If a simple graph has n vertices & $\frac{(n-1)(n-2)}{2} + 1$ edges, it's necessarily connected.

Q) draw a disconnected graph with 6 vertices & 12 edges → not possible.

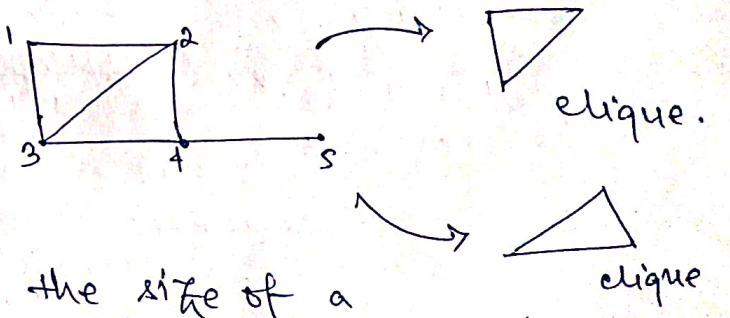


V.V.I.

* clique :- A maximal complete subgraph is called a clique.

V.V.I

the order of the max clique of a graph is called the clique number & denoted by (ω) → Ω



V.V.I

* Girth :- the size of a minimum cycle in a graph is called the girth of the graph. If the graph has no cycle, then its girth is taken as ∞.

Exam

then can't be ≤ 3

V.V+E

Eulerian graph:- A closed walk containing each edge of the graph exactly once called a Eulerian trail & a graph with Eulerian trail is called a Eulerian graph.

Theorem:- A simple connected graph is Eulerian if only if the degree of each vertex of the graph is even

V.V.S

$(d(v) = n \% 2 == 0$

proof:-

(Exam)

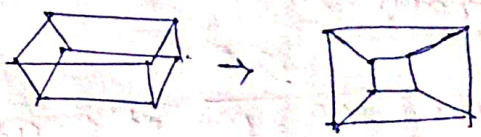
Regular graph:- A graph in which every vertex is of same degree is called a regular graph.

V.V.E

→ degree $(n-1)$ for complete graph

* **Hamiltonian graph:-**

→ world tour → each city - a vertex visit once & reach again at start



Planar graph → redrawing by avoiding edge crossing





$n = 8$
$e = 12$
$f = 6$

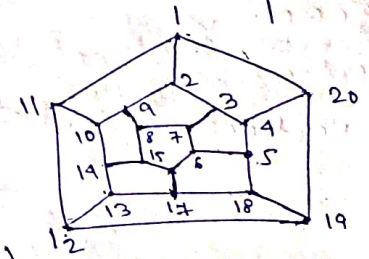
nodes / vertices → edges → faces

$n - e + f = 2$ euler's equation

$n - 30 + 12 = 2$

$n = 20$
$e = 30$
$f = 12$

Do-decahedron



Planar represent^{n} of do-decahedron.

spanning subgraph → all vertices but not all edges.

(cycle) → spanning cycle.

If a graph has spanning subgraph which is cycle → **Hamiltonian graph**

→ there is no necessary & sufficient condition.

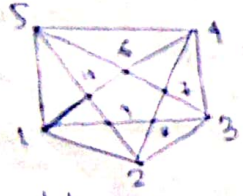
dodecahedron → degree of each vertex = 3

3 Regular graph ←

$(f-2)+3=E$

* If any graph has $d(v)=1$ it's not possible that, it'll be Ham. graph.

⇒



Ham.



Non-Hamiltonian

Hamiltonicity

sufficient conditions:

each ~~any~~ $d(v) \geq 2$.

1) connected graph ✓

$n \rightarrow$ no. of vertices ✓

degree of each vertices $d(v) \geq \left[\frac{n}{2}\right]$

~~this condition~~

** 2) sum of degrees of any pair of adjacent vertices is greater than n.

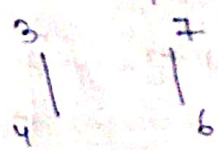
Dirac & Dirac's sufficient & necessary condition

* cut vertex :- A vertex whose removal along with the edges incident to that vertex from a graph increases the no. of components, is called a cut-vertex.

⇒



→

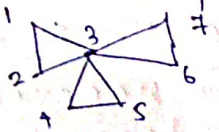


5 → cut vertex,

* Non-separable graph ~~—~~ A graph with no cut-vertex.

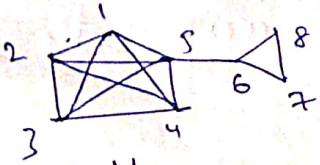


Blocks: Maximal non-separable subgraphs are called blocks.



} 3-blocks.

e.g.



3-blocks / block graph :- If each of the blocks in a graph is a complete graph then its called a block graph.

In a do-decahedron
no single cut vertex —
but certain number of vertices removing

vertex connectivity. { ⩾ 3 ↓ min No. of vertices to be removed

* Min no. of vertex connectivity of a cycle → 2

cut edge | bridge - { If increases no. of components for any removal of edge

→ an edge whose removal ↑es no. of components. & make it disconnected.

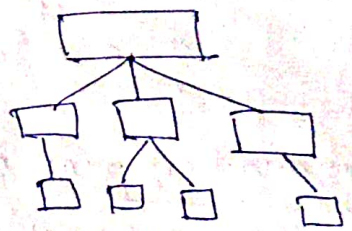
edge connectivity :— the min of edges to be removed simultaneously that ↑es no. of components

Discrete Maths - 15 (18/11/24)

Tree: A tree is a cycle-free connected graph.

Forest: A cycle-free graph is a forest.





$A \subseteq A$

any subset other than ϕ & the set itself is a proper subset of the set.

$S \leq S$ is true ? T

$4 \leq S$ is true ? T.

* $T = \{1, 2\}$

$A \subseteq B$. $S = \{1, 2, 4, -5, 10, 8\}$

$\Rightarrow$ T is a maximal subset of S whose elements does not exceed the value 5

minimal are greater than 3

* minimal ≠ min / infimum / least.
* maximal ≠ max / supremum / greatest.

S is called maximal w.r.t P iff there is no proper superset of S which has the property P.

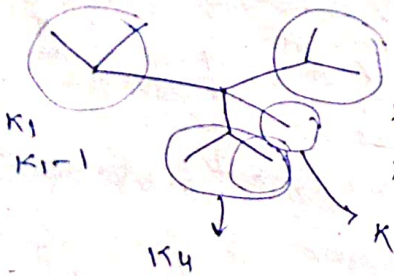
S is a set
P is a property.

* characteristics of a tree :-

1) minimally connected graph.
2) if a tree has n vertices then it has (n-1) edges.

proof → * suppose . it is a tree of $n \le K$. we consider a tree T with $n = K+1$ vertices. it has 'K' edges

No. of edges in ~~every~~ components $K-r$



$K_i < K$

$$\sum_{i=1}^{r}(K_i-1) = \sum_{i=1}^{r} K_i - r = K-r$$

No. of edges $= K-r+r = K$.

3) A tree is a cycle-free graph with n-vertices & (n-1) edges.
4) A tree is a connected graph with n vertices & n-1 edges.
5) A tree has atleast two pendant vertices

· n = vertices · e = n-1.

if $d_i \ge 2$ for $i = 1, 2, \dots n$

then $\sum_{i=1}^{n} d_i \ge 2n$

// handshaking lemma

$2e = 2(n-1)$
$2n-2 \ge 2n$ ✗

if one vertex is $d_i = 1$ other ≥ 2.

$2(n-1) = 2e = \sum d_i \ge 2(n-1)+1$ ✗

* two pendant vertices

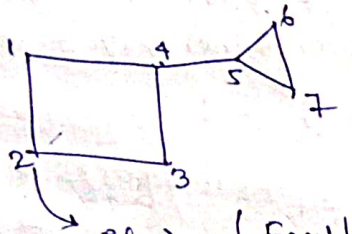
$d_1 = d_2 = 1$

$2(n-1) = 2e = \sum d_i = 2(n-2)+2$ ✓

6) There is a unique path between every pair of vertices of a tree.
(there is no alternative).

* Eccentricity of a vertex :- the distance of the farthest vertex from a vertex v is called the ecc. of a vertex v & is denoted by $e(v)$.

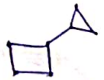
$e(1) = 3$
$e(2) = 4$
$e(3) = 3$
$e(4) = 2$
$e(5) = 3$
$e(6) = 4$
$e(7) = 4$



$e(2)$ → (Must be max.) | Farthest vertex 2 to 6/7)

* Diameter of a graph :-

$$diam(G) = \max_{v \in V(G)} (e(v))$$



$diam(G) = 4$

* Radius of a graph :-

$$rad(G) = \min_{v \in V(G)} e(v).$$

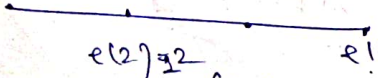
$rad(G) = 2$

* centre of a graph: A vertex whose ecc. is equal to the radius of the graph is called a centre of the graph.

=>

$e(2) = 2$ & rad is also 2
so 2 is the centre of the graph.

$e(1) = 3$, $e(3) = 2$

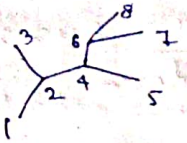


(P_4)

$diam(P_4) = 3$, $rad(P_4) = 2$

* char. of tree:

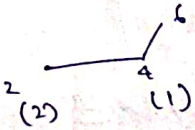
1) A tree has either a single centre or two centres which are adjacent.



* the pendant vertices will have the max ecc.

* After removing the pendant vertices. the remaining vertices will lose the ecc. by 1.

1, 3, 5, 7, 8 are pendant vertices.

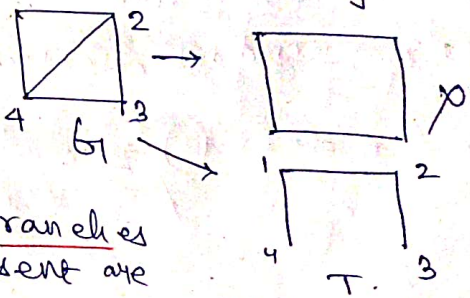


If we keep on removing the vertices by 1. either single vertex will be left or two adjacent vertex.

V.V.I

* Spanning tree :-

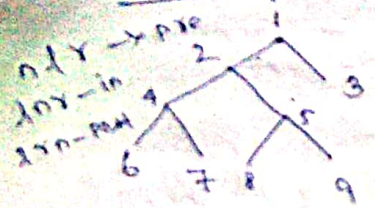
A tree which is a spanning subgraph of a connected graph is called a spanning tree of the graph.



In a spanning tree the edges which are present are called branches & which are not present are called chords / links

In the graph itself not connected then it's not possible to get 'V' spanning subgraph. which is connected, it will be called as spanning forest.

* Binary Tree:-



Traversals

- Preorder:- 124675893
- Inorder:- 647285913
- postorder:- 674895231

* Find a binary tree with 6 vertices)
⇒ impossible.

* is it possible to find a graph with 5 vertices of odd degree—
⇒ impossible

* A binary Tree will always have odd number of vertices

* If a binary tree has n vertices, the no. of pendant vertices is

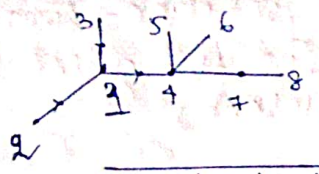
$p = (n+1)/2$.

$1*2 + p \times 1 + (n-p-1) \times 3$
$= 2(n-1)$
$2 + p + 3n - 3p - 3$
$= 2n - 2$
$n - 2p$

Discrete Maths-17 (27/11/24)

* prüfer code of a Tree:-

Any labelled tree on n vertices {1, 2, ---n} can be represented by a sequence of (n-2) integers from {1, 2, ---n} known as prüfer code of a tree.



Tree 'T' with 8 vertices {1, 2 --- 8}.

① ① ④ ④ ④ ⑦
prüfer code
length

1. choose the least labelled pendant vertex & record its unique neighbour.
2. Remove the pendant vertex chosen in step 1 & the edge incident to it.
3. Repeat steps 1 & 2 until a code of length (n-2) is obtained.

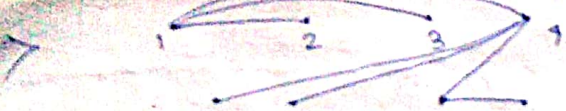
Implementation of Tree by the code:

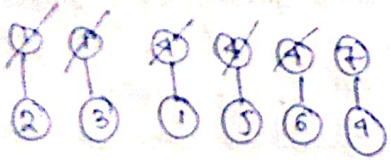
① ① ④ ④ ④ ⑦ → length ⑥

{1, 2, --- 8} → numbers

node = 8 as n.

n-2



P: 

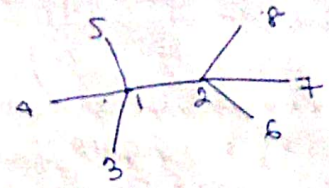
1) choose the least integer from 1 to n which is not in the given code P

2) Draw an edge b/w the integer chosen in step 1 & the first integer in the code P.

3) Remove the first integer from P.

4) Repeat step (1 to 3) until two integers ~~are le~~ not choosen earlier are left to be chosen. Draw an edge b/w those two vertices.

e.g.

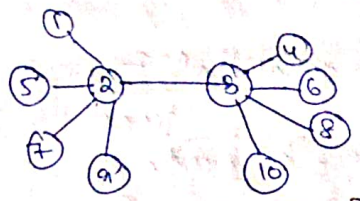


P: (1) (1) (1) (2) (2) (2)

code_length = 6.

e.g.

(2) (3) (4) (3) (2) (3) (2) (3)



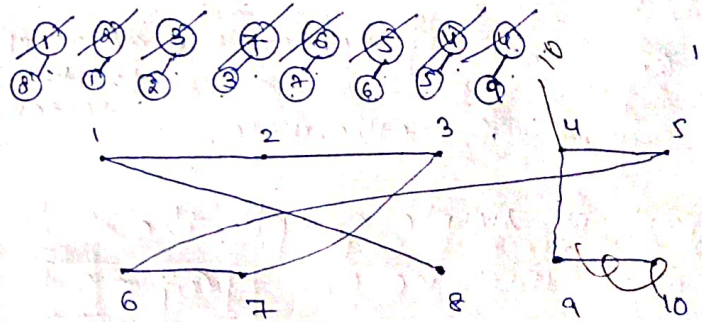


(star) topology

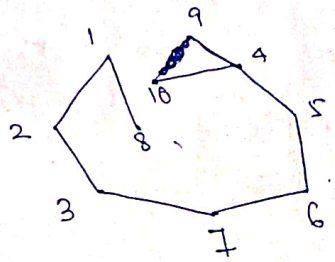
what this geometry is called?

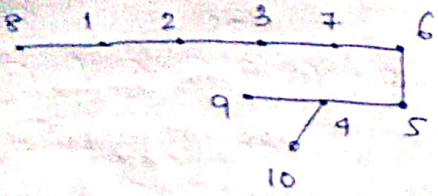
⇒ Ans:- Molecular struc.

* 1 - - - 10



⇒





caterpiller.

No. of spanning trees of K_n.

total n^{n-2} trees

by code

1	1	2	3
1	2		
1	3		
1	4		

*

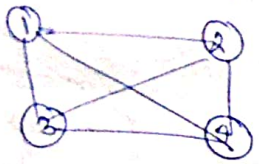


Fig. 1

Adjacency Matrix:

$A = (a_{ij})_{n \times n}$

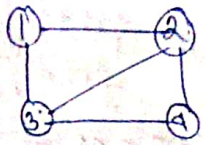
where $a_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to the vertex } j \\ 0 & \text{otherwise.} \end{cases}$

* if the graph is simple graph the diagonal will be 0. $a_{ij} = 0$ where $i = j$.

for Fig. 1.

$$A(K_4) = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}_{4 \times 4}$$

e.g.



$$A(K_4) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

the matrix is symmetric matrix if the graph is undirected.

⇒ if you take the row sum / column sum that'll be your degree of vertex. the corresponding.

⇒ $D = \begin{bmatrix} d_1 & & \bigcirc \\ & d_2 & \\ \bigcirc & \cdots & d_n \end{bmatrix}$

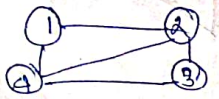
degree matrix is a diagonal matrix

↳ diagonal elems represents the degree of the correspondance vertex.

⇒ Laplacian matrix:- $L = D - A$

degree matrix. Adjacency

e.g.



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Minor a_{ij} → discard ith row, jth col, determinant of the left matrix.

co-factor including sign also.

In (1)

$$M_{01} = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{vmatrix}$$

$$= 3(5) + 1(-2-1) - 1(1+3)$$

$$= 15 - 3 - 4 = 8$$

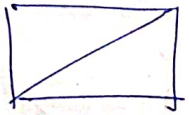
$$M = \begin{vmatrix} -1 & -1 & -1 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = -1(5) + 1(-2) - 1(1)$$

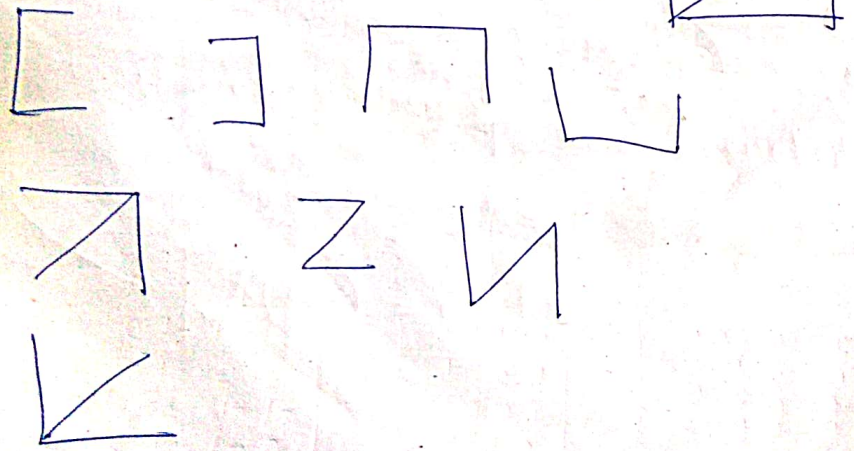
$$= -5 - 2 - 1 = -8$$

$$M_{11} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= 2(6-1) + 1(-2) \cdot = 10 - 2 = 8$$

⇒ 8 spanning tree of





* minor of any element of the laplacian matrix of K_n is of absolute value n^{n-2}.

→ proof. (+10)

→ kirchoff's matrix tree theorem

det of any co-factor (minor) of the L gives the no. of spanning trees of the graph

for a complete graph K_n, no. of spanning trees n^{n-2}

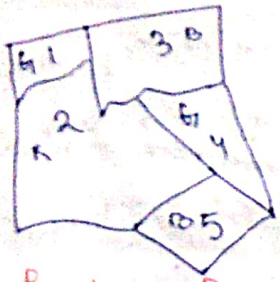
⇒

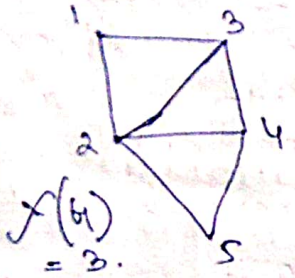
$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}, \quad M_{01} = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 8$$

K_4 : $n=4$, $n^{n-2} = 4^{4-2} = 16$. (multiple of 8). Proved.

* coloring of a graph is a function that maps the set of vertices V of the graph G to a set of colors C, which is subset of the natural nos. N. this mapping must satisfy the condition that if two vertices v_1 & v_2 are adjacent (edge) then the colors assigned to them $f(v_1)$ & $f(v_2)$ must be different.

* coloring of a graph :-





$\chi(G) = 3.$

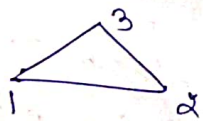
$f(1) = 1$
$f(2) = 2$
$f(3) = 3$
$f(4) = 1$
$f(5) = 3$

defn of coloring of a graph :-

A coloring (proper) of a graph G is a mapping $f: V(G) \to C \subseteq \mathbb{N}$

s.t. $v_1 \sim v_2 \Rightarrow f(v_1) \neq f(v_2)$.

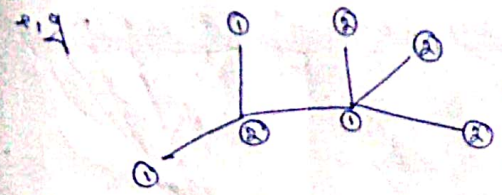
* If a graph is K_3 / C_3 Min no. of colors is req. $= 3$

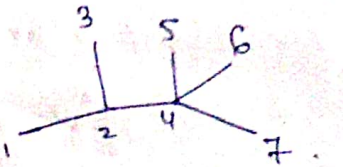


K_3 / C_3

* **chromatic number** : Min. no. of colors req. to color a graph (properly) is called chromatic number of G is denoted by $\chi(G)$.

$\chi(K_n) = n.$





2 colors needed to color the graph.

$\chi(C_n) = 2$ if n is even
$\chi(C_n) = 3$ if n is odd

$\chi(K_n) = n.$ (complete)

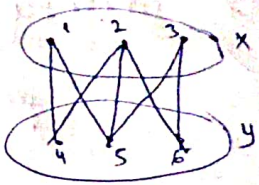
$\chi(T) = 2$ for any tree 'T'.

V.V.I

* Bipartite Graph :-

A Graph 'G' of the vertices can be divided in two Grps Independent sets partitioned such that no two vertices of the same set are adjacent.

If two ind. set are x & y the graph will be denoted as (xy) bipartite graph.



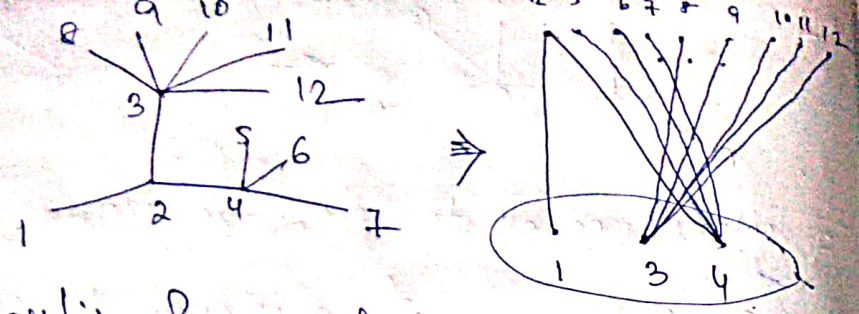
* **Independent set :**
Let $G = (V, E)$ be graph. $S \subseteq V$ is called an indep. set if no two vertices in S are adjacent.

$\Rightarrow$ A bipartite graph is bichromatic

χ(bipartite graph) $= 2$

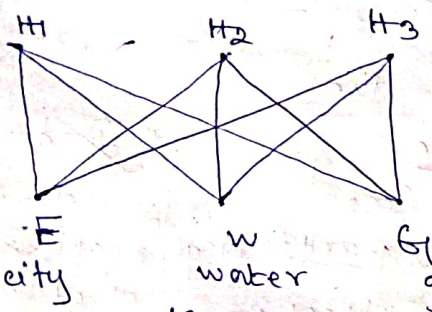
V.V.I

* If a graph contain odd cycle, it can never be bipartite, the converse is true.



* Bi-partite & Bi-chromatic are equivalent, bi-partite ⇔ Bi-chromatic

② utility problem :-



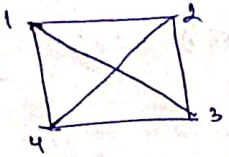
$K_{3,3}$

$K_{m,n}$ → is complete bipartite graph with max no. of edges possible, & the Partite sets are of size m & n respectively.

⇒ No. of edges for $K_{m,n}$ → m*n

*** chromatic polynomial

K_4 :



4! = 24

4 colors of 1,2,3,4 combination

$^4P_4$ arrangements, in such a way we can color the graph.

⇒ if 5 colors are there, the permutation will be:

5 × 4 × 3 × 2 = 120 = 5!

if 6: 6 × 5 × 4 × 3 = 360. (4 nodes).

for, λ colors in K_n graph :-

$$^{\lambda}P_n = \frac{\lfloor \lambda}{\lfloor \lambda - n} = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1).$$

④ chromatic polynomial :- Let G be a graph with n-vertices. Then the no. of ways by which G can be colored using λ or fewer colors is given by a polynomial in λ of degree n, called - of G & is denoted by $P_n(\lambda; G)$

$$P_n(\lambda; K_n) = \lambda(\lambda-1)\cdots(\lambda-n+1).$$

→ λ-(n-1).

n-factors